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The Character of Homogeneous Turbulence

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ABSTRACT

We discuss the turbulent motion of fluid flow by using the renormalization group to solve the stochastic field theory of the velocity field $v_j(\vec{x}, t)$. An "anti-velocity" field is introduced to enable us to give a Lagrangian density for the field theory, and then this density is utilized in the definition of the generating functional for velocity correlation functions. This generating functional contains a stochasticity parameter, a, in such a fashion that when $a \to 0$ the fluid motion is deterministic. This parameter plays the same role as $\hbar$ in the stochastic field theory called quantum field theory. The renormalization group analysis is carried through in detail for a mixing of the fluid which uniformly stirs all modes in wave number and frequency. In such a situation we show that the energy spectrum function $E(k)$ behaves in three dimensions for large k as a constant and for small k as $k^{-\rho}$ where ρ is a universal dynamical index dependent only on the number of space dimensions. It is zero at $D = 4$, and in an expansion about $D = 4$ we find to first order $\rho = -2(4-D)/9$.

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I. INTRODUCTION

Homogeneous turbulence represents an abstraction from the actual more complex turbulent motion encountered in nature. Yet as a mathematical model, often realized in practice to excellent accuracy, it has proven amenable to extensive and profound analysis.[1] Even in the idealization represented by homogeneous turbulence the non-linearities in the Navier-Stokes equation has remained a formidable barrier to the quantitative analysis of turbulent dynamics. Since it is agreed that turbulence is the random motion of a velocity field, $v_j(\vec{x}, t)$, the non-linearities exhibit themselves in the coupling of nth order correlation functions of the velocity to $n + 1^{st}$ order correlations through the inertial term, $v_\alpha \Delta_\alpha v_j$ in the Navier-Stokes equation. The "closure" problem of these velocity correlation functions has remained a central difficulty of the theory of turbulence.[2]

This feature of closure among different orders of correlation functions is a common aspect of stochastic field theories with non-linearities in the governing equations. In the quantum theory of fields as encountered in particle physics[3] and many body physics[4] as well as in classical statistical problems such as phase transitions[5] the equations among the correlation functions fail to close thus making the theories essentially intractable to exact analytical solution.

In such a situation a large variety of approximation techniques have arisen, many of them more or less identical in the various subjects mentioned above though they tend to carry different names. One of the more elegant type of approximation has been to resum the perturbation series in an expansion in the non-linearity to give exact, non-linear equations among the "renormalized" correlation function.[6-9] Then these equations are truncated and solved as well as possible. The "direct interaction" approximation is typical of such a procedure.

This paper treats the stochastic field theory of $v_j(\vec{x}, t)$ by use of the renormalization group.[10] This technique allows one to derive exact constraints on the allowed form of the nth order velocity correlation functions and to extract exact statements about their behavior in both the short distance (large wave number) and long distance (small wave number) regimes. The renormalization group may not be familiar to all workers in the theory of turbulence so a pedagogical introduction to it will be included below.[10] In essence the renormalization group notes that the parameters specifying the non-stochastic theory, for example, the viscosity, ν_o, are replaced by functions of wave number and frequency by the fluctuations in the stochastic theory. One may define renormalized parameters, say ν, by evaluating these functions at some standard wave number, k_N. Then requiring the physical content of the theory to be invariant under changes in k_N, which is arbitrary after all, gives rise to constraints on the allowed form of the correlation functions.

An important example of these constraints is given by the consideration of the energy spectrum function E(k) defined in D space and one time dimension via

$$\frac{1}{D-1}\,\frac{\Gamma(D/2)}{2\pi^{D/2}}\,\frac{E(k)}{k^{D-1}}\left(\delta_{j\ell}-\frac{k_j k_\ell}{k^2}\right) = \int d^D k\, e^{ik\cdot x} \langle v_j(\vec{x}, t) v_\ell(0, \tau)\rangle \quad . \tag{1}$$

In a theory where the medium is uniformly, though randomly, mixed in wave number, the renormalization group predicts that for $k \to 0$, i.e. long wave length turbulence

$$E(k) \underset{k\to 0}{\sim} \nu^2 k_N \left(\frac{k_N}{k}\right)^{3-D+P} \times \text{constant} \quad . \tag{2}$$

where ρ is a universal numerical index which depends only on the number of dimensions of space and not on ν or k_N or the strength of the stirring force. If we introduce the usual viscous energy dissipation parameter

$$\mathscr{E} = 2\nu \int_0^\infty k^2 E(k) dk \tag{3}$$

$$= \nu^3 k_N^{\,4} \times (\text{constant}) \quad , \tag{4}$$

then (2) becomes

$$E(k) \underset{k\to 0}{\sim} \frac{\mathscr{E}^{2/3}}{k_N^{\,5/3}} \left(\frac{k_N}{k}\right)^{3-D+\rho} \times (\text{constant}) \quad . \tag{5}$$

Determining ρ exactly may be tantamount to solving the field theory exactly. Several techniques have been developed to evaluate it approximately.[11] One of these, which we shall explain in detail below, notes that at $D = 4$, $\rho = 0$ and tries an expansion in $\epsilon = 4 - D$. In first order of this expansion we will show $\rho = -2\epsilon/9$ which implies that at $D = 3$

$$E(k) \underset{k\to 0}{\sim} k^{+2/9} \quad . \tag{6}$$

The expansion in ϵ is surely asymptotic, though experience indicates it is very accurate for $\epsilon = 1$.

For large k the form of E(k) is given by the renormalization group to be

$$E(k) \xrightarrow[k\to\infty]{} \frac{\mathscr{E}^{2/3}}{k_N^{\,5/3}} k^{D-3} \left(\text{constant} + 0\left(\frac{1}{k^{4-D}}\right)\right) \tag{7}$$

or at D = 3

$$E(k) \to \text{constant} + O(1/k) \quad . \tag{8}$$

These results indicate that the familiar Kolmogorov behavior

$$E(k) \sim \mathcal{E}^{2/3}/k^{5/3} \tag{9}$$

does not hold precisely for any k when the medium is mixed as described.

The Kolmogorov spectrum is supposed to hold for a turbulent medium predominately mixed at low wave number, so it is not a surprise that it doesn't apply here. In a subsequent paper we shall treat the more physical situation where the medium is mixed in wave number space with the strength $((kL)^2 + 1)^{-\lambda}$, where L is a characteristic length for the mixing and λ is some large power. This mixing spectrum is the same as the one treated here for $k \to 0$, but is cut off at large k; i.e. in the inertial and dissipative regimes. So the results for $k \to 0$ will be the same, and we will explore in detail the interial range. The uniform spectrum treated here will then provide the most easy introduction to the use of the renormalization group as well as illuminating the character of the small k regime.

In the following we begin by a detailed specification of the stochastic field theory appropriate to the non-stochastic Navier-Stokes equation. Then the perturbation series for this field theory is analyzed; the dimensionless expansion parameter is, as usual, the Reynolds number. The special role played by D = 4 will emerge at this point. Next we discuss the renormalization scheme for the stochastic field theory and introduce the renormalization group. After extracting the general statements about velocity correlation functions which follow from the renormalization group, we derive (2) and (7). A discussion of results ends the paper.

II. STOCHASTIC FIELD THEORY FOR TURBULENCE

We want to discuss the stochastic behavior of the velocity field $v_j(\vec{x}, t)$ which satisfies the Navier-Stokes equation

$$\frac{\partial v_j(\vec{x}, t)}{\partial t} = \nu\nabla^2 v_j(\vec{x}, t) - v_\ell(\vec{x}, t)\nabla_\ell v_j(\vec{x}, t) - \frac{1}{\rho}\nabla_j p(\vec{x}, t) + f_j(\vec{x}, t) \tag{10}$$

in the presence of an external force $\vec{f}$. As usual we will take an incompressible fluid

$$\nabla_j v_j(\vec{x}, t) = 0 \tag{11}$$

which will allow us to eliminate the pressure p. Now v_j satisfies (10) in the absence of stochastic behavior, and we want to formulate the stochastic field theory of v_j in such a way that (1) when a certain "stochasticity parameter," called a here, tends to zero, (10) holds precisely, and (2), for any value of a, (10) holds in the mean.

This is accomplished in the following steps. Introduce a Lagrangian density, $\mathscr{L}$, such that the stationarity of the action

$$\int d^D x dt \mathscr{L} \tag{12}$$

yields the equations of motion above. To construct this Lagrange density we must introduce in addition to v_j an "anti-velocity" $\bar{v}_j$ since the Navier-Stokes equation is linear in time derivatives. We will then write for $\mathscr{L}$

$$\begin{aligned} \mathscr{L} &= \tfrac{1}{2}\bar{v}_j \overleftrightarrow{\partial}_t v_j + \nu\nabla_\ell \bar{v}_j \nabla_\ell v_j \\ &- \tfrac{1}{2}[(\Delta_{j\ell}(\nabla)\nabla_n + \Delta_{jn}(\nabla)\nabla_\ell)\bar{v}_j] v_\ell v_n + \mathscr{L}_{\text{Mixing}} \end{aligned} \tag{13}$$

where

$$A \overleftrightarrow{\partial}_t B = A\left(\frac{\partial}{\partial t}B\right) - \left(\frac{\partial A}{\partial t}\right) B \quad , \tag{14}$$

and

$$\Delta_{j\ell}(\nabla) = \delta_{j\ell} - \frac{1}{\nabla^2}\nabla_j\nabla_\ell \tag{15}$$

is a shorthand notation for the operator indicated. $\Delta_{j\ell}$ arises when we eliminate the pressure using $\nabla_j v_j = 0$. Strictly speaking we should enforce the condition $\nabla_j v_j = \nabla_j \bar{v}_j = 0$ by either a Lagrange multiplier in (13) or by replacing $v_j(\vec{x}, t)$ by $\Delta_{j\ell}(\nabla)v_\ell(\vec{x}, t)$ and $\bar{v}_j$ by $\Delta_{j\ell}\bar{v}_\ell$ everywhere. This would significantly complicate the notation and not be very helpful, so we'll use the abbreviated version shown.

The external force term is contained in the mixing Lagrange density. We'll take the external force to be a Gaussian random field with correlation function

$$\langle f_i(\vec{x}, t) f_j(\vec{y}, \tau)\rangle = \Delta_{ij}(\nabla)\,\Gamma(\vec{x} - \vec{y}, t - \tau) \tag{16}$$

and $\mathcal{L}_{\text{Mixing}}$ will be[6,8]

$$\mathcal{L}_{\text{Mixing}} = -\tfrac{1}{2}\bar{v}_j(\vec{x}, t)\int d^D y\, d\tau\, \Delta_{j\ell}(\nabla)\Gamma(\vec{x} - \vec{y}, t - \tau)\bar{v}_\ell(\vec{y}, \tau) \quad . \tag{17}$$

For the greater part of this paper we will discuss a stirring of the fluid which is uniform in wave number and frequency space. So we take

$$\Gamma(\vec{x}, t) = \frac{\Upsilon^2}{4}\delta^D(\vec{x})\,\delta(t) \tag{18}$$

The Lagrange density reads

$$\mathcal{L} = \tfrac{1}{2}\bar{v}_j \overleftrightarrow{\partial}_t v_j + \nu \nabla_\ell \bar{v}_j \nabla_\ell v_j - \frac{\gamma^2}{8} \bar{v}_j \Delta_{j\ell} \bar{v}_\ell$$

$$- \tfrac{1}{2}[(\Delta_{jn}(\nabla)\nabla_\ell + \Delta_{j\ell}(\nabla)\nabla_n)\bar{v}_j] v_\ell v_n \tag{19}$$

At this stage we are still dealing with non-stochastic behavior determined by the Euler-Lagrange equations

$$\frac{d\mathcal{L}}{dv_j} = 0 \quad ; \quad \frac{d\mathcal{L}}{d\bar{v}_j} = 0 \quad , \tag{20}$$

which are just (10) and a similar equation for $\bar{v}_j$. The stochastic nature of the process may be most compactly introduced by giving the generating functional $Z[\eta_j, \bar{\eta}_j]$ for the velocity (and "anti-velocity") time ordered correlation functions

$$< T(v_{j1}(\vec{x}_1, t_1)\ldots v_{jn}(\vec{x}_n, t_n)\bar{v}_{k1}(\vec{y}_1, \tau_1)\ldots\bar{v}_{km}(\vec{y}_m, \tau_m)) > \quad . \tag{21}$$

Z is expressible as a functional integral

$$Z[\eta_j, \bar{\eta}_j] = N \int \prod_{\substack{\vec{x},t \\ j}} dv_j(\vec{x}, t)\delta(\nabla_n v_n(\vec{x}, t)) \prod_{\substack{\vec{y},\tau \\ k}} d\bar{v}_k(\vec{y}, \tau)\delta(\nabla_\ell \bar{v}_\ell(\vec{y}, \tau))$$

$$\times \exp - \frac{1}{a}\int d^D x dt [\mathcal{L}(v_j, \bar{v}_k) + \eta_j(\vec{x}, t)v_j(\vec{x}, t) + \bar{\eta}_k(\vec{x}, t)\bar{v}_k(\vec{x}, t)] \quad , \tag{22}$$

where N guarantees $Z[0, 0] = 1$. The parameter a is the stochasticity variable referred to above. When $a \to 0$, the only field configurations which contribute to $Z[\eta_j, \bar{\eta}_j]$ are those for which the exponent in (22) is stationary

$$\frac{d\mathcal{L}}{dv_j} = -\eta_j \quad \text{and} \quad \frac{d\mathcal{L}}{d\bar{v}_j} = -\bar{\eta}_j \quad , \tag{24}$$

which are the non-stochastic equations with external sources η_j and $\bar{\eta}_j$. Since $Z[\eta_j, \bar{\eta}_j]$ is unchanged when we change the integration variables v_j and $\bar{v}_j$ to $v_j + \delta v_j$ and $\bar{v}_j + \delta\bar{v}_j$ respectively, we find by expanding in δv_j and $\delta\bar{v}_j$

$$< \frac{d\mathscr{L}}{d\bar{v}_j} > = -\eta_j \quad \text{and} < \frac{d\mathscr{L}}{dv_j} > = -\bar{\eta}_j \tag{24}$$

for any a. The mean, expressed here by $<>$, signifies

$$<A(v_j,\bar{v}_j)> = N \int dv_j d\bar{v}_j A(v_j,\bar{v}_j) \exp\left[-\frac{1}{a}\int d^Dx dt \left[\mathscr{L} + \eta_j v_j + \bar{\eta}_j \bar{v}_j\right]\right] \delta(\nabla_j v_j)\delta(\nabla_\ell \bar{v}_\ell) \tag{25}$$

for any functional of v_j and $\bar{v}_j$. Eventually we are interested in $\eta_j = \bar{\eta}_j = 0$ for the correlation functions of v_j and $\bar{v}_j$ alone.

The stochasticity parameter deserves a word of comment. It plays the same role here as does Planck's constant $\hbar$ in quantum theory. We understand this as follows: when $\hbar \to 0$, there are no quantum fluctuations in the system, it is purely classical and deterministic, that is, non-stochastic. In the operator formulation of classical statistical dynamics as discussed in Reference 6, the commutation relations of v_j and $\bar{v}_j$ would be

$$[v_j(\vec{x}, t), \bar{v}_\ell(\vec{y}, t)] = a\ \delta^D(\vec{x} - \vec{y})\,\delta_{j\ell} \tag{26}$$

which again shows the role of a. In quantum theory a is replaced by $\hbar$ in (26). The dimensions of a, as is also true of $\hbar$, are those of the action (12). Without further comment we will most often simply set a = 1 and do our dimensional analysis in those units. If one keeps a throughout, it will count the closed loops appearing in the graphical form of the perturbation theory in Reynolds' number to be developed below.

A useful expression for the Reynolds' number, which we will call g, may be extracted from the Lagrange density (19). g is given as

$$g = \frac{v}{\nu k_N} \quad , \tag{27}$$

where v is the "size" of the velocity $v_j(\vec{x}, t)$ and k_N is some standard $(\text{length})^{-1}$ or wave number. In the situation where the turbulent medium has come to equilibrium the energy input, $\gamma^2 \bar{v}^2$, is essentially balancing the viscous dissipation, $\nu k_N{}^2 v\bar{v}$, so

$$v \simeq \frac{\gamma^2}{\nu k_N{}^2} \bar{v} \quad . \tag{28}$$

From the dimensional analysis of the action, or consideration of the commutation relation (26) we conclude

$$v\bar{v} \simeq k_N{}^D a \quad , \tag{29}$$

leading to

$$v \simeq \frac{\gamma}{\sqrt{\nu}} k_N{}^{\frac{D-2}{2}} \sqrt{a} \tag{30}$$

and

$$g = \frac{\gamma}{\nu^{3/2}} k_N{}^{\frac{D-4}{2}} \sqrt{a} \quad . \tag{31}$$

It is important to note that g becomes independent of k_N at $D = 4$, that is at four space and one time dimension. Whenever the dimensionless parameter characterizing the non-linearities in a stochastic field theory becomes independent of scale, the behavior of the correlation functions becomes quite simple in either the large k or small k limits. Furthermore at that space dimension, the series expansion of correlation functions shows logarithmic divergences and requires infinite renormalizations as occurs in quantum electrodynamics.[3] This fact, on the face of it would seem to have no particular relevance to the real world at $D = 3$ which is our concern. However the scale invariance of the $D = 4$ field theory renders it, in many ways, much more tractable than theories with $D < 4$. Since some form of renormalization will be necessary when we expand in g, it appears convenient to choose that renormalization procedure so as to render the $D = 4$ theory finite. The theory at $D = 3$ is always finite and needs only finite renormalizations. Having chosen this route, we will find it both possible and convenient to expand certain quantities in the $D = 3$ theory about their value at $D = 4$. An example of this is the energy spectrum index ρ discussed in the Introduction. $\rho = 0$ at $D = 4$ and a series in powers of $(4 - D)$ seems sensible.

Clearly the precise method of renormalization must not affect the physical results we derive from our Lagrange density. Indeed it is the invocation of the renormalization group which will be crucial in this by telling us what constraints on the correlation functions must be met to guarantee independence of the renormalization procedure.

III. PERTURBATION THEORY AND RENORMALIZATION FOR VELOCITY CORRELATION FUNCTIONS

In this section we want to give the rules for calculating the velocity correlation functions (21) by expansion of $Z\{\eta_j, \bar{\eta}_j\}$ in powers of γ (or of g). To this end it proves to be useful to rescale the velocity as

$$\chi_j(\vec{x}, t) = 2v_j(\vec{x}, t)/\gamma_0$$

and

$$\bar{\chi}_j(\vec{x}, t) = \frac{\gamma_0}{2}\bar{v}_j(\vec{x}, t) \quad . \tag{32}$$

Then the Lagrange density becomes

$$\mathscr{L}(\chi_j, \bar{\chi}_j) = \tfrac{1}{2}\chi_j \overleftrightarrow{\partial}_t \chi_j + \nu_0 \nabla_\ell \chi_j \nabla_\ell \chi_j$$

$$- \tfrac{1}{2}\bar{\chi}_j \Delta_{j\ell}\bar{\chi}_\ell - \frac{\gamma_0}{4}\left[(\Delta_{jn}(\nabla)\nabla_\ell + \Delta_{j\ell}(\nabla)\nabla_n)\bar{\chi}_j\right]\chi_\ell \chi_n \quad . \tag{33}$$

From this $\mathscr{L}$ and the generating functional (25) we can derive rules for a graphical expression of the terms in the expansion in γ_0 of the correlation function $G^{(n,m)}_{i_1 \ldots i_n, j_1 \ldots j_m}(\vec{k}_1, \omega_1, \ldots \vec{k}_n, \omega_n, \vec{q}_1, \Omega_1, \ldots \vec{q}_m, \Omega_m)$ in wave number, frequency space

$$\delta\left(\sum_{j=1}^{n}\omega_j - \sum_{\ell=1}^{m}\Omega_\ell\right)\delta^D\left(\sum_{j=1}^{n}\vec{k}_j - \sum_{\ell=1}^{m}\vec{q}_\ell\right)G^{(n,m)}_{i_1 \ldots i_n, j_1 \ldots j_m}(\vec{k}_1, \omega_1 \ldots \vec{q}_m, \Omega_m)$$

$$= \int d^D x_1 dt_1 \ldots d^D x_n dt_n d^D y_1 d\tau_1 \ldots d^D y_m d\tau_m \quad \times$$

$$\exp\left[-i(\vec{k}_1 \cdot \vec{x}_1 - \omega_1 t_1 + \ldots + \vec{k}_n \cdot \vec{x}_n - \omega_n t_n) + i(\vec{q}_1 \cdot \vec{y}_1 - \Omega_1 \tau_1 + \ldots + \vec{q}_m \cdot \vec{y}_m - \Omega_m \tau_m)\right]$$

$$\langle T(\chi_{i_1}(\vec{x}_1, t_1) \ldots \chi_{i_n}(\vec{x}_n, t_n)\bar{\chi}_{j_1}(\vec{y}_1, \tau_1) \ldots \bar{\chi}_{j_m}(\vec{y}_m, \tau_m))\rangle \quad . \tag{34}$$

These rules are as follows:

1. There are two kinds of 2nd order correlation functions in zeroeth order in γ_o

$$\text{a)}\ \int d^D x dt\, e^{-i\vec{k}\cdot\vec{x}+i\omega t} \langle T(\chi_j(\vec{x}, t)\chi_\ell(0, 0)\rangle$$

$$\equiv D^0_{j\ell}(\vec{k}, \omega);$$

$$D^0_{j\ell}(\vec{k}, \omega) = \Delta_{j\ell}(\vec{k}) \frac{-1}{(i\omega + \nu_o \vec{k}^2 - \varepsilon)(i\omega - \nu_o \vec{k}^2 - \varepsilon)} \quad , \tag{35}$$

and

$$\text{b)}\ \int d^D x dt\, e^{-i\vec{k}\cdot\vec{x}+i\omega t} \langle T(\chi_j(\vec{x}, t)\bar{\chi}_\ell(0, 0))\rangle \equiv F^0_{j\ell}(\vec{k}, \omega);$$

$$F^0_{j\ell}(\vec{k}, \omega) = \Delta_{j\ell}(\vec{k}) \frac{1}{-i\omega + \nu_o \vec{k}^2 + \varepsilon} \tag{36}$$

where

$$\Delta_{j\ell}(\vec{k}) = \delta_{j\ell} - k_j k_\ell / k^2 \quad , \tag{37}$$

and the limit $\varepsilon \to 0^+$ is understood after all calculations are done. This assures proper retardation. These zeroeth order correlation functions (often called propagators) are represented as in Figure 1. A dotted line represents a $\bar{\chi}$ while a solid line represents a χ. The arrows on the line are necessary because of the structure of the $\bar{\chi}\chi\chi$ term in $\mathscr{L}$.

2. There is one kind of vertex where two solid lines join to a dotted line. This fusion vertex is associated with the factor given in Figure 2.

3. At any given order of γ_o draw all topologically distinct graphs constructed out of D^0, F^0, and the vertex in Figure 2.

4. Integrate $\int d^D q d\omega$ around each closed loop.

5. At each vertex conserve $\vec{q}$ and ω.

6. With each graph associate a weight 1 except for closed loops containing two $D_{j\ell}^{0}$ lines; these have weight ½. See Figure 4.

7. Some graphs will vanish because of the retardation prescription in item 1. For example, the graph in Figure 3 is zero.

These rules may also be derived by considering the averages $\langle\chi\ldots\overline{\chi}\rangle$ directly at $\eta_j = \overline{\eta}_j = 0$ as in (25) and noting that each term in the expansion in γ_o is of the form of powers of χ and $\overline{\chi}$ integrated with $\exp -\tfrac{1}{2}(\chi, \overline{\chi})_j M^{-1}_{j\ell} \begin{pmatrix}\chi\\ \overline{\chi}\end{pmatrix}_\ell$. The inverse of the matrix M^{-1} is composed precisely of the two point correlations D^0 and F^0 when expressed in $\vec{q},\omega$ space.

The lowest non-vanishing corrections to D^0, F^0 and the vertex are given in Figures 4, 5, and 6. Indicated also are the counting factors to be associated with each graph.

In doing the perturbation theory we will modify the "bare" correlation function $D^0_{j\ell}(\vec{k},\omega)$ into the full velocity correlation function $G^{(2,0)}_{oj\ell}(\vec{k},\omega, \gamma_o, \nu_o)$ which is given as a power series in γ_o. Similarly $F^0_{j\ell}$ will become the full $G^{(1,1)}_{oj\ell}(\vec{k}, \omega, \gamma_o, \nu_o)$. Each of these must be proportional to $\Delta_{j\ell}(\vec{k})$ because of the transverse nature of the velocity field, so it is useful to define $\mathscr{G}^{(2,0)}_o(\vec{k}^2, \omega, \gamma_o, \nu_o)$ and $\mathscr{G}^{(1,1)}_o(\vec{k}^2, \omega, \gamma_o, \nu_o)$ by

$$G^{(2,0)}_{oj\ell}(\vec{k},\omega, \gamma_o, \nu_o) = \Delta_{j\ell}(\vec{k})\mathscr{G}^{(2,0)}_o(\vec{k}^2, \omega, \gamma_o, \nu_o) \tag{38}$$

and

$$G^{(1,1)}_{oj\ell}(\vec{k}, \omega, \gamma_o, \nu_o) = \Delta_{j\ell}(\vec{k})\mathscr{G}^{(1,1)}_o(\vec{k}^2, \omega, \gamma_o, \nu_o) \quad . \tag{39}$$

When $\gamma_0 = 0$ we have $\mathscr{G}_0^{(1,1)-1} = -i\omega + \nu_0 k^2$ and the obvious expression for $\mathscr{G}_0^{(2,0)-1}$.

When we have calculated $\mathscr{G}_0^{(2,0)}$ and $\mathscr{G}_0^{(1,1)}$ to as high an order of γ_0 as we desire, the ω and $\vec{k}^2$ behavior will be very complicated. We may continue to parametrize that behavior by ν_0 and γ_0 or we may choose to <u>renormalize</u> the theory by defining effective (or renormalized) parameters ν and γ in some manner. For $D < 4$ all the integrals appearing in the series in γ_0 are finite, except possibly at $\omega_i, \vec{k}_i = 0$ simultaneously, and we are <u>not required</u> to renormalize in this fashion. At $D = 4$ every integral in Figures 4, 5, and 6 is logarithmically divergent and we must find a way to define the theory there.

The central exercise in this paper is to explore the consequences of renormalizing and requiring that the physical results of the theory as embodied in the $G^{(n,m)}$ are independent of how this procedure is carried out. As we will see this path will allow us to derive constraints on the $G^{(n,m)}$. If we were able to solve the theory exactly, these constraints must be identities. In any approximation scheme they will limit our attention to forms of $G^{(n,m)}$ consistent with the general structure of the theory.

The actual method we will choose for defining our renormalized quantities will guarantee that the correlation functions $G^{(n,m)}$ are finite at $D = 4$ when expressed in terms of those renormalized parameters. This prescription will be convenient and practical when we come to extract explicit consequences of the renormalization group.

We adopt the following multiplicative renormalization procedure: Rescale the quantities $\chi_{0j}, \bar{\chi}_{0j}, \nu_0$ and γ_0 by dimensionless factors

$$\chi_{oj}(\vec{x}, t) = Z^{1/2}\chi_j(\vec{x}, t) \tag{40}$$

$$\bar{\chi}_{oj}(\vec{x}, t) = \bar{Z}^{1/2}\bar{\chi}_j(\vec{x}, t) \tag{41}$$

$$\nu_o = Z_\nu^{-1}\nu \tag{42}$$

$$\gamma_o = Z_\gamma^{-1}\gamma \quad . \tag{43}$$

Determine these factors by requiring that the renormalized $\mathscr{G}^{(2,0)}(\vec{k}^2, \omega, \nu, \gamma, \omega_N)$ satisfy

$$\frac{\partial}{\partial\omega}\mathscr{G}^{(1,1)}(\vec{k}^2, \omega, \nu, \gamma, \omega_N)^{-1}\Bigg|_{\substack{\vec{k}^2=0 \\ \omega=\omega_N}} = -i \quad , \tag{44}$$

$$\frac{\partial}{\partial\vec{k}^2}\mathscr{G}^{(1,1)}(\vec{k}^2, \omega, \nu, \gamma, \omega_N)^{-1}\Bigg|_{\substack{\vec{k}^2=0 \\ \omega=\omega_N}} = \nu \quad , \tag{45}$$

$$\frac{\partial}{\partial\omega}\mathscr{G}^{(2,0)}(\vec{k}^2, \omega, \nu, \gamma, \omega_N)^{-1}\Bigg|_{\substack{\vec{k}^2=0 \\ \omega=\omega_N}} = 2\omega_N \quad . \tag{46}$$

Noting that the vertex has the general expression $\Delta_{\alpha j}(k)\Delta_{\beta n}(q_1)\Delta_{\gamma\ell}(q_2)\Gamma_{jn\ell}(\vec{k}, \omega; \vec{q}_1, \omega_1, \vec{q}_2, \omega_2)$ with the lowest order term being

$$\Gamma^0_{jn\ell}(\vec{k}, \omega; \vec{q}_1, \omega_1, \vec{q}_2, \omega_2) = \frac{-i\gamma_o}{2(2\pi)^{\frac{D+1}{2}}}[\delta_{jn}k_\ell + \delta_{j\ell}k_n] \quad , \tag{47}$$

we require of the renormalized $\Gamma_{jn\ell}(\vec{k}, q_1, q_2, \nu, \gamma, \omega_N)$

$$\left.\frac{k_j \Gamma_{jn\ell}(k, q_1, q_2, \nu, \gamma, \omega_N)\delta_{n\ell}}{k^2}\right|_{\substack{\vec{k}^2=\vec{q}_1^{\,2}=\vec{q}_2^{\,2}=0 \\ \omega=2\omega_1=2\omega_2=\omega_N}} = \frac{-i\gamma}{(2\pi)^{\frac{D+1}{2}}} \quad . \tag{48}$$

Since $(\mathcal{G}^{(2,0)})^{-1} = Z(\mathcal{G}_o^{(2,0)})^{-1}$, (44) will determine Z and by construction $Z = 1 + O(\gamma_o^2)$. Then (45) determines Z_ν, and (46), $\bar{Z}$. Finally Z_γ comes from (48).

The indicated procedure is therefore to calculate $\mathcal{G}_o^{(2,0)}$, $\mathcal{G}_o^{(1,1)}$, and $\Gamma^0_{jn\ell}$ to some order in γ_o. Determine the Z–factors using these conditions, and then they in turn define the renormalized correlation functions. The necessity to define only four renormalized quantities comes from the actual structure of the Lagrange density where only four terms in fields appear. It takes an enormous effort to demonstrate that the four conditions we have given are indeed just what one needs to define a finite theory at D = 4. (For the example of quantum electrodynamics, see Reference 3, Chapter 19.) The author has not carried this out in detail but is confident it can be done.

We close this section with an important observation on the field renormalization factors Z and $\bar{Z}$. Because of the momentum dependence of the vertex (Figure 2) or equivalently the derivatives in the non-linear term in $\mathcal{L}$, the functions $(\mathcal{G}_o^{(2,0)})^{-1}$ and $(\mathcal{G}_o^{(1,1)})^{-1}$ can be written

$$(\mathcal{G}_o^{(1,1)})^{-1} = -i\omega + \vec{k}^2(\nu_o - \Sigma_o(\vec{k}^2, \omega)) \tag{49}$$

and

$$(\mathcal{G}_o^{(2,0)})^{-1} = [i\omega + \vec{k}^2(\nu_o - \sigma_o(\vec{k}^2, \omega))][-i\omega + \vec{k}^2(\nu_o - \sigma_o^*(\vec{k}^2, \omega))] \quad . \tag{50}$$

Since Z, $\bar{Z}$ are evaluated by taking derivatives with respect to ω and setting $\omega = \omega_N$ and $\vec{k}^2 = 0$, $\underline{Z = \bar{Z} = 1}$ unless Σ_o or σ_o is singular. In any order of perturbation theory they are regular away from the point $\vec{k}^2 = 0 = \omega$, so it seems quite likely that $Z = \bar{Z} = 1$ identically. This is a useful algebraic simplicity of the given renormalization scheme. Furthermore, since the momentum structure of the vertex is dictated by the solenoidal nature of v_j and $\bar{v}_j$, this renormalization scheme takes maximum advantage of that constraint. One must be cautious, however, not to suppose the limit $\omega_N \to 0$ is innocent. Most of the expressions we write in perturbation theory are singular there (an infrared or long distance singularity) since D^0 and F^0 are singular there. Indeed, the utility of $\omega_N \neq 0$ will come when we vary it in the vicinity of $\omega_N \simeq 0$ to explore the infrared structure of turbulence which is incompletely revealed at any finite order of γ_o.

IV. THE RENORMALIZATION GROUP

We begin this section with some useful dimensional analysis. If we choose units so $a = 1$, the action (12) is dimensionless. Let us assign dimensions of k^{-1} = inverse wave number to position, $\vec{x}$, and ω^{-1} to time, t. Then from $\mathscr{L}$ we learn

$$[\mathscr{L}] = k^D \omega \quad , \tag{51}$$

$$[\chi] = k^{D/2}\omega^{-1/2} \quad , \tag{52}$$

$$[\tilde{\chi}] = \omega^{1/2}k^{D/2} \quad , \tag{53}$$

$$[\nu] = \omega k^{-2} \quad , \tag{54}$$

and

$$[\gamma] = \omega^{3/2}k^{-\left(\frac{D+2}{2}\right)} \quad . \tag{55}$$

where [ ] means dimension of enclosed quantity. From this we learn that using the ingredients of perturbation theory, that is, γ, ν, and ω_N, the only dimensionless parameter is

$$g = \frac{\gamma}{\nu^{\frac{D+2}{4}}}\,\omega_N^{\frac{D-4}{4}} \quad . \tag{56}$$

If we introduce k_N as

$$\omega_N = \nu k_N^{\,2} \quad , \tag{57}$$

then

$$g = (\gamma/\nu^{3/2})k_N^{\frac{D-4}{2}} \quad , \tag{58}$$

which is just the Reynolds' number we defined above.

We may return to the actual velocity correlation functions, called $\phi^{(n,m)}$ from the $G^{(n,m)}$ given above by

$$\phi^{(n,m)}(\vec{k}_i, \omega_i, \nu, g, \omega_N) = \left(\frac{\gamma}{2}\right)^{n-m} G^{(n,m)}(\vec{k}_i, \omega_i, \nu, g, \omega_N) \tag{59}$$

$$= \left(\frac{\gamma}{2}\right)^{n-m} G_o^{(n,m)}(\vec{k}_i, \omega_i, \nu_o, \gamma_o) \quad , \tag{60}$$

where we have used $Z = \bar{Z} = 1$ in (60). {Tensor indices arising from the vector nature of v_j or χ_j are suppressed in this section.} Now $G_o^{(n,m)}$ is the full $\chi_j, \bar{\chi}_j$ correlation function calculated to all orders in γ_o. On the left hand side we have the renormalized velocity correlation. If we change ω_N, $G_o^{(n,m)}$ is unaltered, since it never heard of ω_N, as it was evaluated directly from the series in γ_o. However, both γ and ν must change so that $\gamma^{m-n}\phi^{(n,m)}$ is unchanged. This means

$$\omega_N \frac{d}{d\omega_N} G_0^{(n,m)}(\vec{k}_i, \omega_i, \nu_o, \gamma_o)\Big|_{\nu_o, \gamma_o \text{ fixed}} = 0 \quad , \tag{61}$$

or

$$\left[\omega_N \frac{\partial}{\partial\omega_N} + A(g)\frac{\partial}{\partial g} + B(g)\nu\frac{\partial}{\partial\nu}\right]\left\{\left(g\nu^{\frac{D+2}{4}}\omega_N^{\frac{4-D}{4}}\right)^{m-n} \phi^{(n,m)}(\vec{k}_i, \omega_i, \nu, g\omega_N)\right\} = 0 \, , \tag{62}$$

where

$$A(g) = \omega_N \frac{\partial}{\partial\omega_N} g \Big|_{\nu_o, \gamma_o \text{ fixed}} \tag{63}$$

and

$$B(g) = \frac{\omega_N}{\nu} \frac{\partial}{\partial \omega_N} \nu \Big|_{\nu_o, Y_o \text{ fixed}} \quad . \tag{64}$$

This leads to

$$\left[\omega_N \frac{\partial}{\partial \omega_N} + A(g) \frac{\partial}{\partial g} + B(g)\nu \frac{\partial}{\partial \nu} + C_{n,m}(g)\right] \phi^{(n,m)}(\vec{k}_i, \omega_i, \nu, g, \omega_N) = 0 \tag{65}$$

where

$$C_{n,m}(g) = (m-n)\left[1 - \frac{D}{4} + B(g)\frac{(D+2)}{4} + \frac{A(g)}{g}\right] \quad . \tag{66}$$

Equation (62) is the <u>renormalization group equation.</u>

To cast this equation into a useful form we want to remove the derivative from ω_N to the wave numbers $\vec{k}_i$. To do this first replace ω_N by k_N^2 using $\omega_N = \nu k_N^2$ and note

$$\omega_N \frac{\partial}{\partial \omega_N} = \left[1 - B(g)\right] k_N^2 \frac{\partial}{\partial k_N^2} \quad , \tag{67}$$

so

$$\left[k_N^2 \frac{\partial}{\partial k_N^2} + \frac{A(g)}{1 - B(g)} \frac{\partial}{\partial g} + \frac{B(g)}{1 - B(g)} \nu \frac{\partial}{\partial \nu} + \frac{C_{n,m}(g)}{1 - B(g)}\right] \phi^{(n,m)}(\vec{k}_i \omega_i, \nu, g, k_N^2) = 0 \; . \tag{68}$$

Next use the dimensional analysis above to learn

$$\left[\phi^{(n,m)}\right] = k^{D(1-n) + m - n_\omega} \, 1\text{-}2m \quad , \tag{69}$$

so we may write

$$\phi^{(n,m)}(\vec{k}_i, \omega_i, \nu, g, k_N^2) = \nu^{1-2m} (k_N^2)^{1-\frac{1}{2}(n+3m) + D/2(1-n)} \times$$

$$\times \quad \psi_{n,m}\left(\frac{\omega_i}{\nu k_N^2}, \frac{\vec{k}_i}{k_N}, g\right) \tag{70}$$

where $\psi_{n,m}$ is a dimensionless function. This allows us to express

$$\phi^{(n,m)}(\sqrt{\xi}\, k_i, \omega_i, \nu, g, k_N^2) = \xi^{D/2(1-n) + (m-n)/2}\, \phi^{(n,m)}(\vec{k}_j, \omega_j, \xi\nu, g, (k_N^2)/\xi) . \quad (71)$$

This immediately implies

$$\xi \frac{\xi}{\partial \xi}\, \phi^{(n,m)}(\sqrt{\xi}\, \vec{k}_i, \omega_i, \nu, g, k_N^2)$$

$$= \left(\frac{D(1-n) + m-n}{2} + \nu \frac{\partial}{\partial \nu} - k_N^2 \frac{\partial}{\partial k_N^2} \right) \phi^{(n,m)}(\sqrt{\xi}\, \vec{k}_j, \omega_j, \nu, g, k_N^2) = 0 , \quad (72)$$

and

$$\left[\xi \frac{\partial}{\partial \xi} - \alpha(g) \frac{\partial}{\partial g} - \frac{1}{1-B(g)}\, \nu \frac{\partial}{\partial \nu} - \gamma_{n,m}(g) \right] \phi^{(n,m)}(\sqrt{\xi}\, \vec{k}_j, \omega_j, \nu, g, k_N^2) = 0 , \quad (73)$$

with

$$\alpha(g) = \frac{A(g)}{1 - B(g)} \quad (74)$$

and

$$\gamma_{n,m}(g) = \frac{(m-n)(2-D) + 2D(1-n)}{4} + \frac{m-n}{1-B(g)} \,(1 + B(g)/2 + A(g)/g) \quad (75)$$

Now we have an equation for the variation of the velocity correlation functions with wave number. The solution of (73) is

$$\phi^{(n,m)}(\sqrt{\xi}\, \vec{k}_j, \omega_j, \nu, g, k_N^2) = \phi^{(n,m)}(\vec{k}_j, \omega_j, \bar{\nu}(-\log \xi), \bar{g}(-\log \xi), k_N^2) \times$$

$$\times \exp \int_{-\log \xi}^{0} du\, \gamma_{n,m}(\bar{g}(u)) , \quad (76)$$

where the "running" Reynolds' number $\tilde{g}(u)$ and "running" viscosity satisfy

$$\frac{d\tilde{g}(u)}{du} = -\alpha(\tilde{g}(u)) \quad , \tag{77}$$

and

$$\frac{1}{\tilde{\nu}(u)}\frac{d\tilde{\nu}(u)}{du} = -\frac{1}{1-B(\tilde{g}(u))} \quad , \tag{78}$$

with boundary conditions $\tilde{g}(0) = g$, $\tilde{\nu}(0) = \nu$.

We can use this result to study the $\vec{k}_i$ behavior of the velocity correlations if we are able to deal with the ordinary differential equations (77) and (78). Of these, the equation for $\tilde{g}(u)$, the effective Reynolds number is crucial. If, for example, for $\xi \to 0$, $\tilde{g}(-\log \xi) \to$ small number, then (73) tells us that we may determine $\delta^{(n,m)}$ by a <u>perturbation series in the effective Reynolds' number</u> for small $\vec{k}_i$.

Suppose we calculate $\alpha(g)$ in a perturbation series in g and discover that $\alpha(g)$ has a zero at $g = g_1$ with positive slope

$$\alpha(g) = \alpha_1(g - g_1) \ , \ \alpha_1 > 0 \quad . \tag{79}$$

Then for $\tilde{g}(u) \simeq g_1$ we may solve (77)

$$\tilde{g}(u) = g_1 + (g - g_1)e^{-\alpha_1 u} \tag{80}$$

or

$$\tilde{g}(-\log \xi) = g_1 + (g - g_1)\xi^{\alpha_1} \quad . \tag{81}$$

When $\xi \to 0$, $\tilde{g}(-\log \xi) \to g_1$, regardless of the value of $\tilde{g}(0) = g$. So the low wave number or <u>long distance behavior of turbulence is governed by the zeroes of $\alpha(g)$ with positive slope</u>. Similarly, <u>zeroes of $\alpha(g)$ with negative slope govern the large wave number of short distance behavior of turbulence</u>. The zeroes of $\alpha(g)$ are the key physical objects, not the values of g, the renormalized Reynolds' number, or the value of g_0, the non-stochastic or bare Reynolds' number.

If we knew $\alpha(g)$ perfectly, we would know a great deal indeed about the velocity correlation functions. Probably knowing $\alpha(g)$ is equivalent to solving the full theory; at that point the renormalization group is merely a consistency check on the solution. However, if we know $\alpha(g)$ even in perturbation theory in g <u>and</u> the relevant zeroes g_1 occur at small g, then we have a consistent procedure for determining the behavior of the velocity correlations.

We can now determine the functions $A(g)$ and $B(g)$ in perturbation theory by calculating the graphs in Figures 5 and 6 and from the renormalization conditions in the previous section evaluating Z_ν and Z_γ. The calculation is laborious and follows directly from the graphical rules. We find at D = 4, which will be enough for our purposes

$$\omega_N \frac{\partial}{\partial \omega_N} \log Z_\nu \Big|_{D=4} = -\frac{1}{16}\left(\frac{g}{4\pi}\right)^2 , \tag{82}$$

and

$$\omega_N \frac{\partial}{\partial \omega_N} \log Z_\gamma = \frac{3}{64}\left(\frac{g}{4\pi}\right)^2 . \tag{83}$$

To find A(g) we note

$$g = \frac{\gamma}{\nu^{D+2/4}} \omega_N^{D-4/4} \tag{84}$$

$$= \frac{Z_\gamma}{(Z_\nu)^{D+2/4}} \frac{\gamma_0}{\nu_0^{D+2/4}} \omega_N^{D-4/4} , \tag{85}$$

so

$$A(g) = -\frac{4-D}{4} g + g\left[\omega_N \frac{\partial}{\partial \omega_N} \log Z_\gamma - \frac{D+2}{4} \omega_N \frac{\partial}{\partial \omega_N} \log Z_\nu \right] , \tag{86}$$

so evaluating the term in braces at D = 4 we have for $A(g)$

$$A(g) = -\frac{\epsilon}{4} g + \frac{9}{64} g \left(\frac{g}{4\pi}\right)^2 \; ; \quad \epsilon = 4-D \qquad . \tag{87}$$

If ϵ is small we will find $g \lesssim \sqrt{\epsilon}$ so neglect of higher powers of ϵ or g is accurate in (87). For B(g) we have

$$B(g) = -\frac{1}{16}\left(\frac{g}{4\pi}\right)^2 \qquad . \tag{88}$$

At this stage of approximation $\alpha(g)$ has two zeroes

$$g_1 = 0 \qquad \left.\frac{d\alpha}{dg}\right|_{g_1} = -\frac{\epsilon}{4} \; ; \qquad \text{negative for } D < 4 \quad , \tag{89}$$

and

$$\frac{g_2}{4\pi} = \frac{4}{3}\sqrt{\epsilon} \qquad \left.\frac{d\alpha}{dg}\right|_{g_2} = \frac{\epsilon/2}{1+\epsilon/9} \; ; \qquad \text{positive for } D < 4 \quad . \tag{90}$$

The zero of $\alpha(g)$ at $g_1 = 0$ with negative slope is almost a "kinematic" feature of the dimensional analysis of the perturbation series in γ_0 (or g). It means that <u>for large wave number the effective Reynolds number is zero</u>. More accurately, as $\xi \to \infty$ we find

$$\bar{g}(-\log \xi) \to \lambda \xi^{-\epsilon/4} \tag{91}$$

with λ a constant which may be found from integrating (77), and the effective viscosity

$$\bar{\nu}(-\log \xi) = \nu\xi \qquad . \tag{92}$$

At large wave number the non-linearity is effectively suppressed. To determine the behavior of the $\phi^{(n,m)}$ we must take into account the powers of g which premultiply it. We'll do that below.

The zero of $\alpha(g)$ at $g = g_2$ with positive slope is a less obvious feature of the theory. As $\xi \to 0$ we explore the long wave length structure of $\phi^{(n,m)}(\sqrt{\xi}\, \vec{k}_i,\ldots)$. There $\bar{g}(-\log\, \xi) = g_2$ and

$$\bar{v}(-\log\, \xi) = v\xi^{\,1/1-B(g_2)} \quad . \tag{93}$$

In the next section we will investigate the consequences of these zeroes in $\alpha(g)$.

V. IMPLICATIONS OF THE ZEROES IN $\alpha(g)$

In this section we will explore the consequences for the large and small wave number behavior of the velocity correlation functions $\phi^{(n,m)}$ with special attention to the 2nd order correlation $\phi^{(2,0)}$. First we study the zero with positive slope at $g = g_2$. In an expansion about $D = 4$ we learned above that g_2, or strictly speaking the correct expansion parameter for all of our series $g_2/4\pi$, is small. Here we don't assume it is small, so its value will be calculated by other means. Instead we imagine for now that the renormalized Reynolds' number is precisely g_2. The results we derive this way will be approached as limiting values as $\vec{k}_i \to 0$.

When $g = g_2$, of course $\alpha(g_2) = 0$ and $\bar{g}(-\log\, \xi)$ remains fixed at g_2. This is the origin of the name infrared stable point for g_2. Then we have for $\bar{v}(-\log\, \xi)$ the expression (93) exactly. From the solution (76) of the renormalization group equation it follows that

$$\phi^{(n,m)}(\sqrt{\xi}\, \vec{k}_i,\omega_i,\, v,\, g_2,\, k_N^{\,2}) = \phi^{(n,m)}(\vec{k}_i,\omega_i,\, v\xi^{\,1/1-B_2},\, g_2,\, k_N^{\,2}) \quad \times$$

$$\times \quad \xi^{\,[(m-n)(2-D)+2D(1-n)]\frac{1}{4}+(m-n)/(1-B_2)(1+B_2/2)} \quad , \tag{94}$$

with $B_2 = B(g_2)$. All the effects of the interaction are located in the B_2 dependence of the ξ (or $\vec{k}_i$) dependence The remaining powers of ξ dependence are strictly from dimensional analysis. Using the dimensional analysis result for $\phi^{(n,m)}$ shown in (70) we may write

$$\phi^{(n,m)}(\omega,\omega_i,k,k_i,\nu,g_2,k_N^2) = \nu^{1-2m}(k_N^2)^{[2-n-3m+D(1-n)]\frac{1}{2}} \times$$
$$\times \left(\frac{k^2}{k_N^2}\right)^{\frac{1}{2}[(m-n)(2-D)+2D(1-n)]+1/(1-B_2)[(m-n)(1+B_2/2)+1-2m]} \times$$
$$\times\ F_{n,m}\left(\frac{\omega_i}{\omega},\frac{\vec{k}_i}{k},\frac{\omega}{\nu k_N^2}\left(\frac{k_N^2}{k^2}\right)^{1/1-B_2},g_2\right) \quad , \tag{95}$$

where we have singled out any one of the frequencies and called it ω and any one of the wave numbers and called its length k. $F_{n,m}$ is a dimensionless function of its arguments; tensor indices are still suppressed.

For n = 2, m = 0 (95) gives us a result for

$$\phi_{j\ell}^{(2,0)}(\omega,\vec{k},\nu,g_2,k_N^2) = \Delta_{j\ell}(\vec{k})\,\phi^{(2,0)}(\omega,k^2,\nu,g_2,k_N^2) \tag{96}$$

which reads

$$\phi^{(2,0)}(\omega,k^2,\nu,g_2,k_N^2) = \nu k_N^{-D}\left(\frac{k^2}{k_N^2}\right)^{-2/1-B_2} F\left(\frac{\omega}{\nu k_N^2}\left(\frac{k_N^2}{k^2}\right)^{1/1-B_2},g_2\right). \tag{97}$$

We can use this result in the definition of E(k), Equation (1), to learn

$$E(k) = \frac{(D-1)2\pi^{D/2}}{\Gamma(D/2)}\,k^{D-1}\int d\omega\,\phi^{(2,0)}(\omega,k^2,\nu,g_2,k_N^2) \tag{98}$$

$$= \frac{(D-1)2\pi^{D/2}}{\Gamma(D/2)}\,k^{D-1}\nu^2 k_N^{-D+2}\left(\frac{k^2}{k_N^2}\right)^{-1/1-B_2}\int dx\,F(x,g_2)\ . \tag{99}$$

Write $1/(1-B_2) = 1 + B_2/(1-B_2)$, and we have

$$E(k) = \frac{(D-1)(2\pi)^{D/2}}{\Gamma(D/2)} \nu^2 k_N \left(\frac{k_N}{k}\right)^{3-D+\rho} \int dx F(x, g_2) \tag{100}$$

where

$$\rho = 2B_2/(1-B_2) = -\frac{2\varepsilon}{9} \quad , \tag{101}$$

in an expansion about $D = 4$. This is the result quoted in the introduction.

To find the behavior of $\phi^{(n,m)}$ in the $k \to \infty$ regime, it is convenient to extract the factors of γ^{n-m} in (59) and study the renormalization group equation for $G^{(n,m)}$. This leads to

$$\phi^{(n,m)}(\vec{k}_i, \omega_i, \nu, g, k_N^2) = \gamma^{n-m}(k^2)^{D/4(2-n-m)}(\nu k^2)^{1-(3n+M)/2} \times$$

$$\times \tilde{F}_{n,m}\left(\frac{\omega_i}{\omega}, \frac{\omega}{\nu k^2}, \frac{\vec{k}_i}{k}, g\left(\frac{k}{k_N}\right)^{-\varepsilon/2}\right) \quad , \tag{102}$$

using the same analysis as before. So for $\phi^{(2,0)}$

$$\phi^{(2,0)}(k^2, \omega, \nu, g, k_N^2) = \gamma^2 \nu^{-2} k^{-4} \tilde{F}\left(\frac{\omega}{\nu k^2}, g\left(\frac{k}{k_N}\right)^{-\varepsilon/2}\right) \tag{103}$$

and

$$E(k) = \frac{(D-1)2\pi^{D/2}}{\Gamma(D/2)} \gamma^2 \frac{k^{D-3}}{\nu} \int dx \tilde{F}\left(x, g\left(\frac{k}{k_N}\right)^{-\varepsilon/2}\right) \quad . \tag{104}$$

Since $\tilde{F}(x,g) = f_0(x) + f_1(x)g^2 + \dots$, we have

$$E(k) \underset{k\to\infty}{\sim} k^{D-3}(c_0 + c_1 k^{-\varepsilon} + \dots) \tag{105}$$

and at $D = 3$

$$E(k) \underset{k\to\infty}{\sim} c_0 + \frac{c_1}{k} + \dots \tag{106}$$

with c_0, c_1 some constants. These constants and the function $\bar{F}$ are calculable in principle using perturbation theory to evaluate $G^{(2,0)}$. Furthermore, perturbation theory is accurate as $k \to \infty$ because the effective Reynolds' number behaves as $k^{-\epsilon/4}$.

The spectrum function $E(k)$ is very strongly linked to the behavior of $\Gamma(\vec{k})$, the fourier transform of the mixing function in (16). From Novikov[8] we see that $\mathscr{E}$, the dissipative energy loss

$$\mathscr{E} = \int d^D k\, \Gamma(\vec{k}) \tag{107}$$

so if $\Gamma(k) \sim k^{-2\lambda}$ for large k, then $\lambda > D/2$ for $\mathscr{E}$ to exist. Our example of $\Gamma(k)$ = constant which has been treated at length gives infinite $\mathscr{E}$ because it continually pumps energy in at every wavenumber uniformly. The connection between $E(k)$ and $\Gamma(k)$ for large k is also direct:

$$E(k) \underset{k\to\infty}{\sim} k^{D-3}\Gamma(k) \quad . \tag{108}$$

In a subsequent paper we shall study in detail the mixing force with correlation

$$\Gamma(\vec{k}) = (1 + k^2/m^2)^{-\lambda} \quad , \quad \lambda > D/2 \tag{109}$$

which will give finite $\mathscr{E}$ and a possible regime where the Kolmogorov spectrum is correct. The $k \to 0$ behavior of (100) will not be affected by this change of forcing function.

If for some reason we were interested in turbulent motion at $D = 4$, we could use the renormalization group to evaluate the $k \to 0$ limit of $\phi^{(n,m)}$ but we loose the ability to study the $k \to \infty$ regime since $\alpha(g) = \alpha_1 g^3 + O(g^5)$, $\alpha_1 > 0$ for $D = 4$. Equation (100) for the $k \to 0$ limit of $E(k)$ is changed from $k^{D-3-\rho}$ to $k^{D-3} \times$(logarithms of k/k_N).

VI. SUMMARY AND DISCUSSION

We have analyzed in this paper the stochastic field theory relevant to turbulent motion of a fluid using the renormalization group to provide a non-perturbative tool for studying the effect of the non-linearity in the Navier-Stokes equation. We found it possible to cast the stochastic field theory of the velocity field, $v_j(\vec{x}, t)$, into a familiar functional integral form by introducing an "anti" velocity field, $\tilde{v}_j(\vec{x}, t)$, and a stochasticity parameter, a, measuring the deviation from deterministic behavior of the fluid. We did not discuss the stochasticity parameter at any length in this work, but it seems quite plausible that it determines the presence and importance of turbulence in fluid motion. Certainly the Reynolds' number which governs the importance of the non-linear, intertial terms is significant, however, if $a \to 0$, then no fluctuations, i.e. no turbulent motions, are present.

From the generating functional for velocity correlation functions we derived a perturbation theory in the Reynolds' number for those correlation functions, and used the renormalization group to provide a summation technique for that perturbation theory. This enabled us to find the large and small wave number behavior of the many point velocity correlation functions. Particular interest has been focused on the two point velocity correlation $\langle v_j(\vec{x}, t) v_\ell(\vec{y}, \tau) \rangle$, and we exhibited the behavior of it in some detail. Indeed we found that for the theory with uniform mixing in wave number and frequency space that the energy spectrum function $E(k)$ commonly defined in turbulence theory behaves, in three space dimensions, as a constant for large k (see the comments after Eq. (107)) and as $k^{-\rho}$ with ρ a small, negative number for small k. In an expansion about $D = 4$

dimensions, explained in the text, we found $\rho = -2(4-D)/9$, so $\rho = -2/9$ in three dimensions.

A key issue in the theory as presented here is the nature of the driving force for turbulence. It enters the Lagrangian density as a term[6]

$$\mathscr{L}_{\text{Mixing}}(v_j, \bar{v}_\ell) = -\tfrac{1}{2}\bar{v}_j(\vec{x}, t)\int d^D y\, d\tau\, \Gamma_{j\ell}(\vec{x}-\vec{y}, t-\tau)\bar{v}_\ell(\vec{y}, \tau)$$

when the external force in the Navier-Stokes equation is taken to be a Gaussian random field with zero mean and

$$\langle f_j(\vec{x}, t) f_\ell(0, 0)\rangle = \Gamma_{j\ell}(\vec{x}, t) \quad .$$

In this paper we studied the forcing function

$$\Gamma_{j\ell}(\vec{x}, t) = \frac{Y^2}{4}\delta_{j\ell}(\nabla)\,\delta^D(\vec{x})\delta(t)$$

partly from the point of view of simplicity, partly because it seems like a useful point to begin the renormalization group analysis of turbulent motion. Another interesting forcing function would be one which turned on at some time t_0 and turned off at some time t_1. The study of the correlation functions at large $t > t_1$ would illuminate the issues involved in the decay of turbulence. This is likely to be a more practical question than the ones studied here.

Several future projects are suggested by the techniques developed here: 1. Use the renormalization group to study the frequency dependence of the velocity correlation functions. This is straightforward using the dimensional analysis and renormalization group equations discussed above. 2. Use the techniques developed in high energy physics[13] to derive the functions $\dot{\gamma}_{n,m}$ and $F_{n,m}$ near the infrared stable zeroes of $\alpha(g)$. This provides one with the remaining available information on the velocity correlation functions. 3. There is one index, which we called ρ, which determines the $k \to 0$ behavior of all velocity correlation functions. We studied it in an expansion about $D = 4$ space dimensions and to first order found $\rho = -2(4-D)/9$ Such indices may be studied by the equivalent of high temperature expansions as employed in solid state physics. These expansions are straightforward in concept albeit tedious in practice. They are usually remarkably accurate. 4. Finally some attention should be paid to the possibility of experimentally determining a value for the stochasticity parameter a which will enter in a non-trivial fashion into the correlation functions. Several of these projects are now in progress and will be reported on in future articles.

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FIGURE CAPTIONS

Fig. 1: The two point velocity correlation functions (propagators) in the absence of non-linearities. These are used in constructing the perturbation series in Reynolds' number. $\Delta_{j\ell}(k) = \delta_{j\ell} - k_j k_\ell / k^2$.

Fig. 2: The fusion vertex for the non-linearity in the Navier-Stokes equation.

Fig. 3: An example of a perturbation theory graph which is zero because of the retardation in the function F^0 in Figure 1.

Fig. 4: The lowest order terms in the Reynolds' number expansion of the velocity-velocity correlation function.

Fig. 5: The lowest order terms in the Reynolds' number expansion of the velocity–anti-velocity correlation function.

Fig. 6: The lowest order terms in the Reynolds' number expansion of the fusion vertex.

Substitution for paragraph at bottom of page 18:

At this stage an aside is in order. By looking at the time dependence of the decay of homogeneous, isotropic turbulence in the final stages of decay one can learn directly about $\Gamma_M(k^2 = 0)$. There are two competing hypotheses about the behavior of $\Gamma_M(0)$. One is given by Batchelor,[4] Section 5.4, where he argues that the analyticity of the velocity correlation function at k = 0 requires

$$\Gamma_M(k^2) \sim k^2 \tag{75}$$

near $k^2 = 0$. This has been criticized in detail by Saffmann[9] who argues instead that the analyticity assumption is more properly made about the vorticity correlation function. Then one has $\Gamma_M(0)$ finite and, furthermore, an invariant of the motion. An additional argument in favor of Saffman's conjecture is that $\Gamma_M(0) \neq 0$ would imply, for long times into the decay period when the degrees of freedom of the fluid had time to come to equilibrium after whatever mixing had occurred, that the energy spectrum E(k) behaves as k^{D-1} which one expects from equipartition. It is important to note that there is a difference between E(k, t) in non-stationary turbulence and E(k) in the stationary case. It is E(k, t) which for long times after the mixing behaves at $k \to 0$ as k^{D-1} when $\Gamma_M(0) \neq 0$. E(k) has an additional factor of k^{-2} and behaves as $E(k) \sim k^{D-3}$ when $\Gamma_M(0) \neq 0$. For generality, however, it is easy enough to consider a behavior like (75). Then N = 1 at D = 3, and $\tilde{g}(u) \to \infty$ as $u \to \infty$ or $\xi \to 0$.

Add to references

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possible disagreement with that. The issue, then, is the behavior in the intermediate regime where k is large compared to 0 or k_o, but still outside the deep dissipation regime. Here the interpolating formulae derived in Section V of this paper are the tool to explore this region.

As to the behavior of $\tilde{g}$ changing as R_o changes, I cannot agree. I recommend the referee explore the field theory of a scalar field with $\lambda_o\phi^4$ coupling in D dimensions. The nature of perturbation theory in λ_o changes at D = 4, regardless of the size of λ_o. For D < 4 two phases of the theory are possible. One is connected to perturbation theory and has a dissipation region where as $k \to \infty$, $\lambda_{effective} \to 0$. The other has $\lambda_{effective} \to \infty$ as $k \to \infty$. For $D \geq 4$ only one phase exists. The presence of two phases is not dependent on the size of λ_o. The turbulence problem is much the same. Here, however, we are fortunate in having a physical boundary condition to choose the appropriate phase for D < 4. That boundary condition is the existence of a dissipation region where $\nu\nabla^2 v_j$ dominates $\vec{v}\,\nabla v_j$ and the effective Reynolds number goes to zero.

(2) On rereading the paragraph beginning on the bottom of p. 18 I can see how it should be rewritten for clarity. I enclose an altered paragraph to address that.

I think this should make it clear that my preference for $\Gamma_M(0) \neq 0$ has a sound physical basis.

One last comment which pertains to the referee's statement about E(k) decreasing as $\Gamma_M(k)$. For very large k, i.e. in the deep dissipation region that is what physically one would expect some transport by $\vec{v}\cdot\nabla v_j$ has become unimportant. For large k but still less than η^{-1} there is a combination of effects consisting of energy transport by $\vec{v}\cdot\nabla v_j$ and decreased input due to the fall off of Γ_M. It seems to me possible, though not yet demonstrated that a balance yielding a Kolmogorov spectrum could arise, though I do not expect it in a neat analytic sense.

Reply to the Referee on "The Behavior of Homogeneous Turbulence Mixed at Long Wavelengths"

I appreciate the long and careful review given by the referee of my paper. I will try to address the two important points he raises: (1) a question about the behavior of the effective coupling as one varies the bare Reynolds number, and (2) the phrasing on p. 17-19 about $\Gamma_M(0)$.

(1) The renormalized Reynolds number, g, is an infinite series in g_o, the bare or unrenormalized Reynolds number. If the function A(g) has a zero with positive slope at $g = g_1$ and a zero with negative slope at $g = 0$; e.g.

$$A(g) = -\frac{\varepsilon}{4} g + a g^3 \quad , \quad a > 0 \; ,$$

as in the theory of turbulence, then one can solve for the $g(g_o)$ relation by using the boundary condition that when $g_o \to 0$, $g \to 0$. The relation, good at the same level as A(g), is

$$g^2 = \frac{g_o^{\,2}}{1 + g_o^{\,2}/g_1^{\,2}} \quad .$$

From this one sees that as the bare Reynolds number g_o ranges over $0 \le g_o < \infty$, g ranges from zero only up to g_1. The effective coupling $\tilde{g}$ reflects this behavior. As $k \to \infty$, if $g \to 0$ for any g_o, it goes to zero for every g_o.

I believe there is an important physical point to be made, and perhaps that is what the referee is driving at. $k \to \infty$ literally means leaving the inertial range and moving into the deep dissipation range where $k \gg \eta^{-1} = (\varepsilon/\nu^3)^{1/4}$. In that range, whatever the bare Reynolds number for $D < 4$, $\nu\nabla^2 v_j$ will dominate the inertial term. Perhaps I am wrong, but I don't see a

In first paragraph on p. 21 change first sentence to:

For the circumstances mentioned above $\Gamma_M(k^2) \propto k^2$ for small k, N = 1, and we see that $\tilde{G}(u)$ is zero at both ends of the wave number spectrum for D = 3.